

Lagrangian Mechanics on Quaternion Kähler Manifolds

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Abstract

The aim of this study is to introduce quaternion Kähler analogue of Lagrangian mechanics. Finally, the geometric and physical results related to quaternion Kähler dynamical systems are also presented.

Keywords: Quaternion Kähler geometry, Lagrangian Mechanics.

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1 Introduction

It is well-known that modern differential geometry explains explicitly the dynamics of Lagrangians. Therefore, we say that if Q is an m -dimensional configuration manifold and $L : TQ \rightarrow \mathbf{R}$ is a regular Lagrangian function, then there is a unique vector field ξ on TQ such that dynamics equations is given by

$$i_\xi \Phi_L = dE_L \tag{1}$$

where Φ_L indicates the symplectic form. The triple (TQ, Φ_L, ξ) is called *Lagrangian system* on the tangent bundle TQ .

In literature, there are a lot of studies about Lagrangian mechanics, formalisms, systems and equations [1, 2] and there in. There are real, complex, paracomplex and other analogues. It is possible to produce different analogous in different spaces. Finding new dynamics equations is both a new expansion and contribution to science to explain physical events.

Quaternions were invented by Sir William Rowan Hamilton as an extension to the complex numbers. Hamilton's defining relation is most succinctly written as:

$$i^2 = j^2 = k^2 = ijk = -1 \tag{2}$$

If it is compared to the calculus of vectors, quaternions have slipped into the realm of obscurity. They do however still find use in the computation of rotations. A lot of physical laws in classical, relativistic, and quantum mechanics can be written pleasantly by means of quaternions. Some physicists hope they will find deeper understanding of the universe by restating basic principles in terms of quaternion algebra. It is well-known that quaternions are useful for representing rotations in both quantum and classical mechanics [3].

In this study, equations related to Lagrangian mechanical systems on quaternion Kähler manifold have been presented.

2 Preliminaries

Throughout this paper, all mathematical objects and mappings are assumed to be smooth, i.e. infinitely differentiable and Einstein convention of summarizing is adopted. $\mathcal{F}(M)$, $\chi(M)$ and $\Lambda^1(M)$ denote the set of functions on M , the set of vector fields on M and the set of 1-forms on M , respectively.

2.1 Quaternion Kähler Manifolds

Let M be an n -dimensional manifold with a 3-dimensional vector bundle V consisting of tensors of type (1,1) over M satisfying condition as follows:

(a) In any coordinate neighborhood U of M , there exists a local basis $\{F, G, H\}$ of V such that

$$F^2 = -I, G^2 = -I, H^2 = -I \quad (3)$$

$$GH = -HG = F, HF = -FH = G, FG = -GF = H.$$

Where I denotes the identity tensor of type (1,1) in M . $\{F, G, H\}$ is called a canonical local basis of the bundle V in U . Then V is called an almost quaternion structure in M . The pair (M, V) denotes an almost quaternion manifold with V . An almost quaternion manifold M is of dimension $n = 4m$ ($m \geq 1$). In any almost quaternion manifold (M, V) , there is a Riemannian

metric tensor field g such that

$$g(\phi X, Y) + g(X, \phi Y) = 0 \quad (4)$$

for any cross-section ϕ and any vector fields X, Y of M . An almost quaternion structure V with such a Riemannian metric g is called an almost quaternion metric structure. A manifold M with an almost quaternion metric structure $\{g, V\}$ is called an almost quaternion metric manifold. The triple (M, g, V) denotes an almost quaternion metric manifold. Let $\{F, G, H\}$ be a canonical local basis of V an almost quaternion manifold (M, g, V) . Since each of F, G and H is almost Hermitian structure with respect to g metric, taking

$$\Phi(X, Y) = g(FX, Y), \quad \Psi(X, Y) = g(GX, Y), \quad \Theta(X, Y) = g(HX, Y) \quad (5)$$

for any vector fields X and Y , we see that Φ, Ψ and Θ are local 2-forms.

Suppose that the Riemannian connection ∇ of (M, g, V) satisfies conditions as follows:

(b) If ϕ is a cross-section (local or global) of the bundle V , then $V_X \phi$ is also a cross-section of V , X being an arbitrary vector field in M . From (3) we see that the condition (b) is equivalent to condition as follows:

(b') If F, G, H is a canonical local basis of V , then

$$\nabla_X F = r(X)G - q(X)H, \quad \nabla_X G = -r(X)F + p(X)H, \quad \nabla_X H = q(X)F - p(X)G \quad (6)$$

for any vector field X , where p, q and r are certain local 1-forms. If an almost quaternion metric manifold M satisfies the condition (b) or (b'), then M is called a quaternion Kähler manifold and an almost quaternion structure of M is called a quaternion Kähler structure. [4]

Let $\{x_i, x_{n+i}, x_{2n+i}, x_{3n+i}\}$, $i = \overline{1, n}$ be a real coordinate system on a neighborhood U of M , and let $\left\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_{n+i}}, \frac{\partial}{\partial x_{2n+i}}, \frac{\partial}{\partial x_{3n+i}} \right\}$ and $\{dx_i, dx_{n+i}, dx_{2n+i}, dx_{3n+i}\}$ be natural bases over R

of the tangent space $T(M)$ and the cotangent space $T^*(M)$ of M , respectively. Considering [5], then the following expression can be obtained

$$\begin{aligned} F\left(\frac{\partial}{\partial x_i}\right) &= \frac{\partial}{\partial x_{n+i}}, F\left(\frac{\partial}{\partial x_{n+i}}\right) = -\frac{\partial}{\partial x_i}, F\left(\frac{\partial}{\partial x_{2n+i}}\right) = \frac{\partial}{\partial x_{3n+i}}, F\left(\frac{\partial}{\partial x_{3n+i}}\right) = -\frac{\partial}{\partial x_{2n+i}} \\ G\left(\frac{\partial}{\partial x_i}\right) &= \frac{\partial}{\partial x_{2n+i}}, G\left(\frac{\partial}{\partial x_{n+i}}\right) = -\frac{\partial}{\partial x_{3n+i}}, G\left(\frac{\partial}{\partial x_{2n+i}}\right) = -\frac{\partial}{\partial x_i}, G\left(\frac{\partial}{\partial x_{3n+i}}\right) = \frac{\partial}{\partial x_{n+i}} \\ H\left(\frac{\partial}{\partial x_i}\right) &= \frac{\partial}{\partial x_{3n+i}}, H\left(\frac{\partial}{\partial x_{n+i}}\right) = \frac{\partial}{\partial x_{2n+i}}, H\left(\frac{\partial}{\partial x_{2n+i}}\right) = -\frac{\partial}{\partial x_{n+i}}, H\left(\frac{\partial}{\partial x_{3n+i}}\right) = -\frac{\partial}{\partial x_i} \end{aligned}$$

3 Lagrangian Mechanics

In this section, we obtain Euler-Lagrange equations for quantum and classical mechanics by means of a canonical local basis $\{F, G, H\}$ of V on quaternion Kähler manifold (M, V) .

Firstly, let F take a local basis component on the quaternion Kähler manifold (M, V) , and $\{x_i, x_{n+i}, x_{2n+i}, x_{3n+i}\}$ be its coordinate functions. Let semispray be the vector field ξ determined by

$$\xi = X^i \frac{\partial}{\partial x_i} + X^{n+i} \frac{\partial}{\partial x_{n+i}} + X^{2n+i} \frac{\partial}{\partial x_{2n+i}} + X^{3n+i} \frac{\partial}{\partial x_{3n+i}}, \quad (7)$$

where $X^i = \dot{x}_i, X^{n+i} = \dot{x}_{n+i}, X^{2n+i} = \dot{x}_{2n+i}, X^{3n+i} = \dot{x}_{3n+i}$ and the dot indicates the derivative with respect to time t . The vector field defined by

$$V_F = F(\xi) = X^i \frac{\partial}{\partial x_{n+i}} - X^{n+i} \frac{\partial}{\partial x_i} + X^{2n+i} \frac{\partial}{\partial x_{3n+i}} - X^{3n+i} \frac{\partial}{\partial x_{2n+i}} \quad (8)$$

is called *Liouville vector field* on the quaternion Kähler manifold (M, V) . The maps given by $T, P : M \rightarrow R$ such that $T = \frac{1}{2}m_i(\dot{x}_i^2 + \dot{x}_{n+i}^2 + \dot{x}_{2n+i}^2 + \dot{x}_{3n+i}^2), P = m_i gh$ are called *the kinetic energy* and *the potential energy of the system*, respectively. Here m_i, g and h stand for mass of a mechanical system having m particles, the gravity acceleration and distance to the origin of a

mechanical system on the quaternion Kähler manifold (M, V) , respectively. Then $L : M \rightarrow R$ is a map that satisfies the conditions; i) $L = T - P$ is a *Lagrangian function*, ii) the function given by $E_L^F = V_F(L) - L$, is *energy function*.

The operator i_F induced by F and given by

$$i_F \omega(X_1, X_2, \dots, X_r) = \sum_{i=1}^r \omega(X_1, \dots, F X_i, \dots, X_r), \quad (9)$$

is said to be *vertical derivation*, where $\omega \in \wedge^r M$, $X_i \in \chi(M)$. The *vertical differentiation* d_F is defined by

$$d_F = [i_F, d] = i_F d - d i_F \quad (10)$$

where d is the usual exterior derivation. For F , the closed Kähler form is the closed 2-form given by $\Phi_L^F = -dd_F L$ such that

$$d_F = \frac{\partial}{\partial x_{n+i}} dx_i - \frac{\partial}{\partial x_i} dx_{n+i} + \frac{\partial}{\partial x_{3n+i}} dx_{2n+i} - \frac{\partial}{\partial x_{2n+i}} dx_{3n+i} : \mathcal{F}(M) \rightarrow \wedge^1 M. \quad (11)$$

Then

$$\begin{aligned} \Phi_L^F = & -\frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j \wedge dx_i + \frac{\partial^2 L}{\partial x_j \partial x_i} dx_j \wedge dx_{n+i} - \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j \wedge dx_{2n+i} \\ & + \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j \wedge dx_{3n+i} - \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+j} \wedge dx_i + \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{n+j} \wedge dx_{n+i} \\ & - \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{n+j} \wedge dx_{2n+i} + \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} \wedge dx_{3n+i} - \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{2n+j} \wedge dx_i \\ & + \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{2n+j} \wedge dx_{n+i} - \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{2n+j} \wedge dx_{2n+i} + \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{2n+j} \wedge dx_{3n+i} \\ & - \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{3n+j} \wedge dx_i + \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{3n+j} \wedge dx_{n+i} - \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{3n+j} \wedge dx_{2n+i} \\ & + \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+j} \wedge dx_{3n+i} \end{aligned} \quad (12)$$

Let ξ be the second order differential equation given by **Eq. (1)** and

$$\begin{aligned}
i_\xi \Phi_L^F = & -X^i \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta_i^j dx_i + X^i \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j + X^i \frac{\partial^2 L}{\partial x_j \partial x_i} \delta_i^j dx_{n+i} \\
& -X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_i} dx_j - X^i \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} \delta_i^j dx_{2n+i} + X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j + X^i \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} \delta_i^j dx_{3n+i} \\
& -X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j - X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} \delta_{n+i}^{n+j} dx_i + X^i \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+j} \\
& +X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} \delta_{n+i}^{n+j} dx_{n+i} - X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{n+j} - X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} \delta_{n+i}^{n+j} dx_{2n+i} \\
& +X^{2n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{n+j} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} \delta_{n+i}^{n+j} dx_{3n+i} - X^{3n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} \\
& -X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} \delta_{2n+i}^{2n+j} dx_i + X^i \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{2n+j} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} \delta_{2n+i}^{2n+j} dx_{n+i} \\
& -X^{n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{2n+j} - X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} \delta_{2n+i}^{2n+j} dx_{2n+i} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{2n+j} \\
& +X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} \delta_{2n+i}^{2n+j} dx_{3n+i} - X^{3n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{2n+j} - X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} \delta_{3n+i}^{3n+j} dx_i \\
& +X^i \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{3n+j} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} \delta_{3n+i}^{3n+j} dx_{n+i} - X^{n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{3n+j} \\
& -X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} \delta_{3n+i}^{3n+j} dx_{2n+i} + X^{2n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{3n+j} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} \delta_{3n+i}^{3n+j} dx_{3n+i} \\
& -X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+j}.
\end{aligned} \tag{13}$$

Since the closed quaternion Kähler form Φ_L^F on (M, V) is the symplectic structure, it is found

$$E_L^F = V_F(L) - L = X^i \frac{\partial L}{\partial x_{n+i}} - X^{n+i} \frac{\partial L}{\partial x_i} + X^{2n+i} \frac{\partial L}{\partial x_{3n+i}} - X^{3n+i} \frac{\partial L}{\partial x_{2n+i}} - L$$

and hence

$$\begin{aligned}
dE_L^F = & X^i \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_i} dx_j + X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j - X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j \\
& + X^i \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+j} - X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{n+j} + X^{2n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{n+j} - X^{3n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} \\
& + X^i \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{2n+j} - X^{n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{2n+j} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{2n+j} - X^{3n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{2n+j} \\
& + X^i \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{3n+j} - X^{n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{3n+j} + X^{2n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{3n+j} - X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+j} \\
& - \frac{\partial L}{\partial x_j} dx_j - \frac{\partial L}{\partial x_{n+j}} dx_{n+j} - \frac{\partial L}{\partial x_{2n+j}} dx_{2n+j} - \frac{\partial L}{\partial x_{3n+j}} dx_{3n+j}
\end{aligned} \tag{14}$$

With the use of **Eq.** (1), the following expressions can be obtained:

$$\begin{aligned}
& -X^i \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j + X^i \frac{\partial^2 L}{\partial x_j \partial x_i} dx_{n+j} - X^i \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_{2n+j} + X^i \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_{3n+j} \\
& - X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_j + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{n+j} - X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{2n+j} \\
& + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{3n+j} - X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_j + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{n+j} \\
& - X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{2n+j} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{3n+j} - X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_j \\
& + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{n+j} - X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{2n+j} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+j} \\
& + \frac{\partial L}{\partial x_j} dx_j + \frac{\partial L}{\partial x_{n+j}} dx_{n+j} + \frac{\partial L}{\partial x_{2n+j}} dx_{2n+j} + \frac{\partial L}{\partial x_{3n+j}} dx_{3n+j} = 0
\end{aligned} \tag{15}$$

If a curve denoted by $\alpha : R \rightarrow M$ is considered to be an integral curve of ξ , then we calculate the following equation:

$$\begin{aligned}
& -X^i \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_j - X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_j - X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_j \\
& + X^i \frac{\partial^2 L}{\partial x_j \partial x_i} dx_{n+j} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{n+j} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{n+j} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{n+j} \\
& - X^i \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_{2n+j} - X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{2n+j} - X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{2n+j} - X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{2n+j} \\
& + X^i \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_{3n+j} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{3n+j} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{3n+j} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+j} \\
& + \frac{\partial L}{\partial x_j} dx_j + \frac{\partial L}{\partial x_{n+j}} dx_{n+j} + \frac{\partial L}{\partial x_{2n+j}} dx_{2n+j} + \frac{\partial L}{\partial x_{3n+j}} dx_{3n+j} = 0,
\end{aligned} \tag{16}$$

alternatively

$$\begin{aligned}
& -[X^i \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}}] dx_j + \frac{\partial L}{\partial x_j} dx_j \\
& + [X^i \frac{\partial^2 L}{\partial x_j \partial x_i} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i}] dx_{n+j} + \frac{\partial L}{\partial x_{n+j}} dx_{n+j} \\
& - [X^i \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}}] dx_{2n+j} + \frac{\partial L}{\partial x_{2n+j}} dx_{2n+j} \\
& + [X^i \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}}] dx_{3n+j} + \frac{\partial L}{\partial x_{3n+j}} dx_{3n+j} = 0.
\end{aligned} \tag{17}$$

Then we obtain the equations

$$\begin{aligned}
\frac{\partial}{\partial t} \left(\frac{\partial L}{\partial x_i} \right) + \frac{\partial L}{\partial x_{n+i}} &= 0, \quad \frac{\partial}{\partial t} \left(\frac{\partial L}{\partial x_{n+i}} \right) - \frac{\partial L}{\partial x_i} = 0, \\
\frac{\partial}{\partial t} \left(\frac{\partial L}{\partial x_{2n+i}} \right) + \frac{\partial L}{\partial x_{3n+i}} &= 0, \quad \frac{\partial}{\partial t} \left(\frac{\partial L}{\partial x_{3n+i}} \right) - \frac{\partial L}{\partial x_{2n+i}} = 0,
\end{aligned} \tag{18}$$

such that the equations obtained in **Eq.** (18) are said to be *Euler-Lagrange equations* structured on quaternion Kähler manifold (M, V) by means of Φ_L^F and thus the triple (M, Φ_L^F, ξ) is said to be a *mechanical system* on quaternion Kähler manifold (M, V) .

Secondly, we find Euler-Lagrange equations for quantum and classical mechanics by means of Φ_L^G on quaternion Kähler manifold (M, V) .

Consider G be another local basis component on the quaternion Kähler manifold (M, V) .

Let ξ take as in **Eq.** (7). In the case, the vector field given by

$$V_G = G(\xi) = X^i \frac{\partial}{\partial x_{2n+i}} - X^{n+i} \frac{\partial}{\partial x_{3n+i}} - X^{2n+i} \frac{\partial}{\partial x_i} + X^{3n+i} \frac{\partial}{\partial x_{n+i}} \tag{19}$$

is *Liouville vector field* on the quaternion Kähler manifold (M, V) . The function given by

$E_L^G = V_G(L) - L$ is *energy function*. Then the operator i_G induced by G and denoted by

$$i_G \omega(X_1, X_2, \dots, X_r) = \sum_{i=1}^r \omega(X_1, \dots, G X_i, \dots, X_r) \tag{20}$$

is *vertical derivation*, where $\omega \in \wedge^r M$, $X_i \in \chi(M)$. The *vertical differentiation* d_G are defined

by

$$d_G = [i_G, d] = i_G d - di_G \quad (21)$$

where d is the usual exterior derivation. Since taking into considering G , the closed Kähler form is the closed 2-form given by $\Phi_L^G = -dd_G L$ such that

$$d_G = \frac{\partial}{\partial x_{2n+i}} dx_i - \frac{\partial}{\partial x_{3n+i}} dx_{n+i} - \frac{\partial}{\partial x_i} dx_{2n+i} + \frac{\partial}{\partial x_{n+i}} dx_{3n+i} : \mathcal{F}(M) \rightarrow \wedge^1 M. \quad (22)$$

Then we have

$$\begin{aligned} \Phi_L^G = & -\frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j \wedge dx_i + \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j \wedge dx_{n+i} + \frac{\partial^2 L}{\partial x_j \partial x_i} dx_j \wedge dx_{2n+i} \\ & -\frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j \wedge dx_{3n+i} - \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} \wedge dx_i + \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{n+j} \wedge dx_{n+i} \\ & +\frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{n+j} \wedge dx_{2n+i} - \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+j} \wedge dx_{3n+i} - \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{2n+j} \wedge dx_i \\ & +\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{2n+j} \wedge dx_{n+i} + \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{2n+j} \wedge dx_{2n+i} - \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{2n+j} \wedge dx_{3n+i} \\ & -\frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+j} \wedge dx_i + \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{3n+j} \wedge dx_{n+i} + \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{3n+j} \wedge dx_{2n+i} \\ & -\frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{3n+j} \wedge dx_{3n+i}. \end{aligned} \quad (23)$$

Let ξ be differential equation yielding **Eq.** (1) and, then it follows

$$\begin{aligned}
i_\xi \Phi_L^G = & -X^i \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} \delta_i^j dx_i + X^i \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j + X^i \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} \delta_i^j dx_{n+i} \\
& -X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j + X^i \frac{\partial^2 L}{\partial x_j \partial x_i} \delta_i^j dx_{2n+i} - X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_i} dx_j - X^i \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta_i^j dx_{3n+i} \\
& + X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} \delta_{n+i}^{n+j} dx_i + X^i \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} \\
& + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} \delta_{n+i}^{n+j} dx_{n+i} - X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{n+j} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} \delta_{n+i}^{n+j} dx_{2n+i} \\
& - X^{2n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{n+j} - X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} \delta_{n+i}^{n+j} dx_{3n+i} + X^{3n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+j} \\
& - X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} \delta_{2n+i}^{2n+j} dx_i + X^i \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{2n+j} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} \delta_{2n+i}^{2n+j} dx_{n+i} \\
& - X^{n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{2n+j} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} \delta_{2n+i}^{2n+j} dx_{2n+i} - X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{2n+j} \\
& - X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} \delta_{2n+i}^{2n+j} dx_{3n+i} + X^{3n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{2n+j} - X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} \delta_{3n+i}^{3n+j} dx_i \\
& + X^i \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+j} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} \delta_{3n+i}^{3n+j} dx_{n+i} - X^{n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{3n+j} \\
& + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} \delta_{3n+i}^{3n+j} dx_{2n+i} - X^{2n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{3n+j} - X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} \delta_{3n+i}^{3n+j} dx_{3n+i} \\
& + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{3n+j}
\end{aligned} \tag{24}$$

Since the closed Kähler form Φ_L^G on M is the symplectic structure, it is gotten

$$E_L^G = V_G(L) - L = X^i \frac{\partial L}{\partial x_{2n+i}} - X^{n+i} \frac{\partial L}{\partial x_{3n+i}} - X^{2n+i} \frac{\partial L}{\partial x_i} + X^{3n+i} \frac{\partial L}{\partial x_{n+i}} - L \tag{25}$$

and hence

$$\begin{aligned}
dE_L^G = & X^i \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j - X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j - X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_i} dx_j + X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j \\
& + X^i \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} - X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{n+j} - X^{2n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{n+j} + X^{3n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+j} \\
& + X^i \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{2n+j} - X^{n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{2n+j} - X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{2n+j} + X^{3n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{2n+j} \\
& + X^i \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+j} - X^{n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{3n+j} - X^{2n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{3n+j} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{3n+j} \\
& - \frac{\partial L}{\partial x_j} dx_j - \frac{\partial L}{\partial x_{n+j}} dx_{n+j} - \frac{\partial L}{\partial x_{2n+j}} dx_{2n+j} - \frac{\partial L}{\partial x_{3n+j}} dx_{3n+j}
\end{aligned} \tag{26}$$

By means of **Eq. (1)**, we calculate

$$\begin{aligned}
& -X^i \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j + X^i \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_{n+j} + X^i \frac{\partial^2 L}{\partial x_j \partial x_i} dx_{2n+j} - X^i \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+j} \\
& -X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_j + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{n+j} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{2n+j} \\
& -X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{3n+j} - X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_j + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{n+j} \\
& + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{2n+j} - X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{3n+j} - X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_j \\
& + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{n+j} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{2n+j} - X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{3n+j} \\
& + \frac{\partial L}{\partial x_j} dx_j + \frac{\partial L}{\partial x_{n+j}} dx_{n+j} + \frac{\partial L}{\partial x_{2n+j}} dx_{2n+j} + \frac{\partial L}{\partial x_{3n+j}} dx_{3n+j} = 0
\end{aligned} \tag{27}$$

If a curve, defined by $\alpha : R \rightarrow M$, is an integral curve of ξ , then we obtain

$$\begin{aligned}
& -X^i \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j - X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_j - X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_j - X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_j \\
& + X^i \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_{n+j} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{n+j} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{n+j} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{n+j} \\
& + X^i \frac{\partial^2 L}{\partial x_j \partial x_i} dx_{2n+j} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{2n+j} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{2n+j} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{2n+j} \\
& - X^i \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+j} - X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{3n+j} - X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{3n+j} - X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{3n+j} \\
& + \frac{\partial L}{\partial x_j} dx_j + \frac{\partial L}{\partial x_{n+j}} dx_{n+j} + \frac{\partial L}{\partial x_{2n+j}} dx_{2n+j} + \frac{\partial L}{\partial x_{3n+j}} dx_{3n+j} = 0
\end{aligned} \tag{28}$$

or

$$\begin{aligned}
& -[X^i \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}}] dx_j + \frac{\partial L}{\partial x_j} dx_j \\
& + [X^i \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}}] dx_{n+j} + \frac{\partial L}{\partial x_{n+j}} dx_{n+j} \\
& + [X^i \frac{\partial^2 L}{\partial x_j \partial x_i} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i}] dx_{2n+j} + \frac{\partial L}{\partial x_{2n+j}} dx_{2n+j} \\
& - [X^i \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}}] dx_{3n+j} + \frac{\partial L}{\partial x_{3n+j}} dx_{3n+j} = 0
\end{aligned} \tag{29}$$

Then the equations are found:

$$\begin{aligned}
& \frac{\partial}{\partial t} \left(\frac{\partial L}{\partial x_i} \right) + \frac{\partial L}{\partial x_{2n+i}} = 0, \quad \frac{\partial}{\partial t} \left(\frac{\partial L}{\partial x_{n+i}} \right) - \frac{\partial L}{\partial x_{3n+i}} = 0, \\
& \frac{\partial}{\partial t} \left(\frac{\partial L}{\partial x_{2n+i}} \right) - \frac{\partial L}{\partial x_i} = 0, \quad \frac{\partial}{\partial t} \left(\frac{\partial L}{\partial x_{3n+i}} \right) + \frac{\partial L}{\partial x_{n+i}} = 0.
\end{aligned} \tag{30}$$

Thus the equations obtained in **Eq.** (30) are called *Euler-Lagrange equations* structured by means of Φ_L^G on quaternion Kähler manifold (M, V) and thus the triple (M, Φ_L^G, ξ) can be called to be a *mechanical system* on quaternion Kähler manifold (M, V) .

Thirdly, we introduce Euler-Lagrange equations for quantum and classical mechanics by means of Φ_L^H on quaternion Kähler manifold (M, V) .

Let H be a local basis on the quaternion Kähler manifold (M, V) . Let semispray ξ give as in **Eq.**(7). Therefore, *Liouville vector field* on the quaternion Kähler manifold (M, V) is the vector field given by

$$V_H = H(\xi) = X^i \frac{\partial}{\partial x_{3n+i}} + X^{n+i} \frac{\partial}{\partial x_{2n+i}} - X^{2n+i} \frac{\partial}{\partial x_{n+i}} - X^{3n+i} \frac{\partial}{\partial x_i}. \quad (31)$$

The function given by $E_L^H = V_H(L) - L$ is *energy function*. The function i_H induced by H and shown by

$$i_H \omega(X_1, X_2, \dots, X_r) = \sum_{i=1}^r \omega(X_1, \dots, H X_i, \dots, X_r), \quad (32)$$

is said to be *vertical derivation*, where $\omega \in \wedge^r M$, $X_i \in \chi(M)$. The *vertical differentiation* d_H is denoted by

$$d_H = [i_H, d] = i_H d - d i_H, \quad (33)$$

where d is the usual exterior derivation. Considering H , the closed Kähler form is the closed 2-form given by $\Phi_L^H = -d d_H L$ such that

$$d_H = \frac{\partial}{\partial x_{3n+i}} dx_i + \frac{\partial}{\partial x_{2n+i}} dx_{n+i} - \frac{\partial}{\partial x_{n+i}} dx_{2n+i} - \frac{\partial}{\partial x_i} dx_{3n+i} : \mathcal{F}(M) \rightarrow \wedge^1 M \quad (34)$$

Then we get

$$\begin{aligned}
\Phi_L^H = & -\frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j \wedge dx_i - \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j \wedge dx_{n+i} + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j \wedge dx_{2n+i} \\
& + \frac{\partial^2 L}{\partial x_j \partial x_i} dx_j \wedge dx_{3n+i} - \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{n+j} \wedge dx_i - \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} \wedge dx_{n+i} \\
& + \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+j} \wedge dx_{2n+i} + \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{n+j} \wedge dx_{3n+i} - \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{2n+j} \wedge dx_i \\
& - \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{2n+j} \wedge dx_{n+i} + \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{2n+j} \wedge dx_{2n+i} + \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{2n+j} \wedge dx_{3n+i} \\
& - \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{3n+j} \wedge dx_i - \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+j} \wedge dx_{n+i} + \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{3n+j} \wedge dx_{2n+i} \\
& + \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{3n+j} \wedge dx_{3n+i}
\end{aligned} \tag{35}$$

Let ξ be the semispray given by **Eq. (1)** and, then we find

$$\begin{aligned}
i_\xi \Phi_L^H = & -X^i \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} \delta_i^j dx_i + X^i \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j - X^i \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} \delta_i^j dx_{n+i} \\
& + X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j + X^i \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta_i^j dx_{2n+i} - X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j + X^i \frac{\partial^2 L}{\partial x_j \partial x_i} \delta_i^j dx_{3n+i} \\
& - X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_i} dx_j - X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} \delta_{n+i}^{n+j} dx_i + X^i \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{n+j} \\
& - X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} \delta_{n+i}^{n+j} dx_{n+i} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} \delta_{n+i}^{n+j} dx_{2n+i} \\
& - X^{2n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+j} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} \delta_{n+i}^{n+j} dx_{3n+i} - X^{3n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{n+j} \\
& - X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} \delta_{2n+i}^{2n+j} dx_i + X^i \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{2n+j} - X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} \delta_{2n+i}^{2n+j} dx_{n+i} \\
& + X^{n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{2n+j} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} \delta_{2n+i}^{2n+j} dx_{2n+i} - X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{2n+j} \\
& + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} \delta_{2n+i}^{2n+j} dx_{3n+i} - X^{3n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{2n+j} - X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} \delta_{3n+i}^{3n+j} dx_i \\
& + X^i \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{3n+j} - X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} \delta_{3n+i}^{3n+j} dx_{n+i} + X^{n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+j} \\
& + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} \delta_{3n+i}^{3n+j} dx_{2n+i} - X^{2n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{3n+j} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} \delta_{3n+i}^{3n+j} dx_{3n+i} \\
& - X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{3n+j}
\end{aligned} \tag{36}$$

Since the closed quaternion Kähler form Φ_L^H on M is the symplectic structure, it is found

$$E_L^H = V_H(L) - L = X^i \frac{\partial L}{\partial x_{3n+i}} + X^{n+i} \frac{\partial L}{\partial x_{2n+i}} - X^{2n+i} \frac{\partial L}{\partial x_{n+i}} - X^{3n+i} \frac{\partial L}{\partial x_i} - L. \tag{37}$$

Hence we have

$$\begin{aligned}
dE_L^H = & X^i \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j + X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j - X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_i} dx_j \\
& X^i \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{n+j} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} - X^{2n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+j} - X^{3n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{n+j} \\
& X^i \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{2n+j} + X^{n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{2n+j} - X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{2n+j} - X^{3n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{2n+j} \\
& X^i \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{3n+j} + X^{n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+j} - X^{2n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{3n+j} - X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{3n+j} \\
& - \frac{\partial L}{\partial x_j} dx_j - \frac{\partial L}{\partial x_{n+j}} dx_{n+j} - \frac{\partial L}{\partial x_{2n+j}} dx_{2n+j} - \frac{\partial L}{\partial x_{3n+j}} dx_{3n+j}.
\end{aligned} \tag{38}$$

Using **Eq. (1)**, we calculate the following expression:

$$\begin{aligned}
& -X^i \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j - X^i \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_{n+j} + X^i \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{2n+j} + X^i \frac{\partial^2 L}{\partial x_j \partial x_i} dx_{3n+j} \\
& -X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_j - X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{2n+j} \\
& +X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{3n+j} - X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_j - X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{n+j} \\
& +X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{2n+j} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{3n+j} - X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_j \\
& -X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{n+j} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{2n+j} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{3n+j} \\
& + \frac{\partial L}{\partial x_j} dx_j + \frac{\partial L}{\partial x_{n+j}} dx_{n+j} + \frac{\partial L}{\partial x_{2n+j}} dx_{2n+j} + \frac{\partial L}{\partial x_{3n+j}} dx_{3n+j} = 0.
\end{aligned} \tag{39}$$

If a curve, defined by $\alpha : R \rightarrow M$, is considered to be an integral curve of ξ , then it holds the equation as follows:

$$\begin{aligned}
& -X^i \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j - X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_j - X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_j - X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_j \\
& -X^i \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_{n+j} - X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} - X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{n+j} - X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{n+j} \\
& +X^i \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{2n+j} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{2n+j} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{2n+j} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{2n+j} \\
& +X^i \frac{\partial^2 L}{\partial x_j \partial x_i} dx_{3n+j} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{3n+j} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{3n+j} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{3n+j} \\
& + \frac{\partial L}{\partial x_j} dx_j + \frac{\partial L}{\partial x_{n+j}} dx_{n+j} + \frac{\partial L}{\partial x_{2n+j}} dx_{2n+j} + \frac{\partial L}{\partial x_{3n+j}} dx_{3n+j} = 0.
\end{aligned} \tag{40}$$

or alternatively

$$\begin{aligned}
& -[X^i \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}}] dx_j + \frac{\partial L}{\partial x_j} dx_j \\
& -[X^i \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}}] dx_{n+j} + \frac{\partial L}{\partial x_{n+j}} dx_{n+j} \\
& +[X^i \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}}] dx_{2n+j} + \frac{\partial L}{\partial x_{2n+j}} dx_{2n+j} \\
& +[X^i \frac{\partial^2 L}{\partial x_j \partial x_i} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i}] dx_{3n+j} + \frac{\partial L}{\partial x_{3n+j}} dx_{3n+j} = 0
\end{aligned} \tag{41}$$

Then we find the equations

$$\begin{aligned}
& \frac{\partial}{\partial t} \left(\frac{\partial L}{\partial x_i} \right) + \frac{\partial L}{\partial x_{3n+i}} = 0, \quad \frac{\partial}{\partial t} \left(\frac{\partial L}{\partial x_{n+i}} \right) + \frac{\partial L}{\partial x_{2n+i}} = 0, \\
& \frac{\partial}{\partial t} \left(\frac{\partial L}{\partial x_{2n+i}} \right) - \frac{\partial L}{\partial x_{n+i}} = 0, \quad \frac{\partial}{\partial t} \left(\frac{\partial L}{\partial x_{3n+i}} \right) - \frac{\partial L}{\partial x_i} = 0.
\end{aligned} \tag{42}$$

Thus the equations given in **Eq.** (42) infer *Euler-Lagrange equations* structured by means of Φ_L^H on quaternion Kähler manifold (M, V) and thus the triple (M, Φ_L^H, ξ) is said to be a *mechanical system* on quaternion Kähler manifold (M, V) .

4 Conclusion

From above, Lagrangian mechanics has intrinsically been described taking into account a canonical local basis $\{F, G, H\}$ of V on quaternion Kähler manifold (M, V) .

The paths of semispray ξ on the quaternion Kähler manifold are the solutions Euler–Lagrange equations raised in (18), (30) and (42), and obtained by a canonical local basis $\{F, G, H\}$ of vector bundle V on quaternion Kähler manifold (M, V) . One can be proved that these equations are very important to explain the rotational spatial mechanics problems.

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